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On Schlömilch series representation for the transverse electric multiple scattering by an infinite grating of insulating dielectric circular cylinders at oblique incidence

Ömer Kavaklıoğlu

Division of Electrophysics Research and Department of Pure and Applied Mathematics, Department of Physics, Electronics and Electrical Engineering, Girne American University at Karaoğlanoğlu, Kuzey Kıbris Türk Cumhuriyeti, Girne (Mersin-10) İçel, Turkey

E-mail: kavaklioglu_omer@yahoo.com

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Abstract

Elementary function representations of *Schlömilch series* introduced by Twersky (Twersky V 1961 *Arch. Ration. Mech. Anal.* **8** 323–32) are used to construct the exact analytical expressions for the classical electromagnetic problem of transverse electric multiple scattering by an infinite array of insulating dielectric circular cylinders at oblique incidence.

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1. Introduction

Twersky solved the problem of diffraction of plane electromagnetic waves at normal incidence on a infinite grating of insulating dielectric circular cylinders as long ago as 1956 [2, 3]. Since then many other authors have studied a configuration of greater relevance to the problem. For instance, Shestopalov *et al* [4, 5] have reiterated and improved his results and Bogdanov *et al* [6] have constructed an algorithm for the problem of diffraction of a plane electromagnetic wave, incident arbitrarily on a periodic array of infinitely long dielectric rods of circular cross section. They presented the relations between the main diffraction characteristics of the array and its parameters. In addition, Bogdanov *et al* [7–9] have extended their solution for arbitrary incidence to various configurations including coaxial dielectric cylinders and coaxial metal–dielectric cylinders, and they studied the dependence of the transmission coefficient of the incident wave on various parameters of the problem. In a more recent study, Bogdanov *et al* [10] examined both theoretically and experimentally the interaction of a plane electromagnetic wave with an array made up of a periodic arrangement of dielectric cylindrical columns. They investigated the normal-incidence case for a wave whose electric field vector is parallel

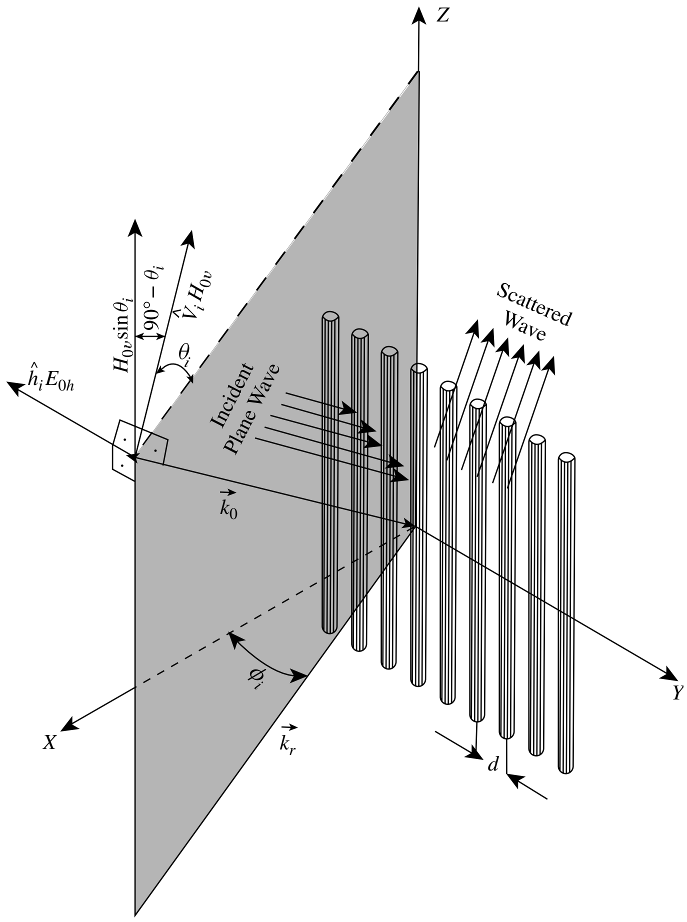

Figure 1. The infinite grating of infinitely long dielectric circular cylinders.

to the cylinder axes. The investigation is carried out in the resonance region of the array for three types of structure that differ in their fill coefficients.

In these previous works cited above the solutions for the fields do not cover the most generalized case of oblique incidence, even though the grating is irradiated from the side of the x-axis by an incident plane-polarized electromagnetic wave at an arbitrary angle to the x-axis, whereas in the generalized oblique incidence solution presented in our work the direction of the incident plane wave makes an arbitrary oblique angle θ_i with the positive z-axis, as indicated in figure 1. As far as can be ascertained by this author, the most generalized case of scattering by an infinite grating of circular dielectric cylinders for an obliquely incident plane E-polarized wave has only recently been published by Kavaklıoğlu [11, 12]. The solutions for the exact representations of the external and internal fields corresponding to the vertical polarization case were obtained by direct application of the separation-of-variables technique and the addition theorem for cylindrical waves. This technique leads to two systems of simultaneous linear equations of infinite order in coupled form for the scattering coefficients associated with the external electric and magnetic fields [11]. Employing the Sommerfeld integral representation of the Hankel function and exploiting the Poisson summation formula in separation-of-variables solution the exact expressions for the diffracted electric and magnetic field intensities were derived in terms of the diffraction angles of the infinite grating of circular dielectric cylinders for an obliquely incident transverse magnetic plane wave [12].

The purpose of this paper is to present the most generalized exact solution for 'the multiple scattering of transverse electric plane waves at oblique incidence by an infinite array of insulating dielectric circular cylinders' in terms of *Schlömilch series* and special functions. The arbitrarily polarized incident plane electromagnetic wave shown in figure 1 can be decomposed into two components yielding two different modes of polarization. The component for which the incident electric field $\vec{E}^{\mathrm{inc}}$ has no component parallel to the cylinders will be treated in this paper. This is the horizontal polarization, which is also known as the transverse electric (TE) mode. For this case, the incident H-field has a component that is parallel to all the cylinders. For the TE mode, $\hat{h}_i$ is the unit vector associated with the horizontal polarization case that is parallel to the x–y plane and does not have any component parallel to the cylinders. The fact that 'the incident H-field has a component that is parallel to all the cylinders' does not mean that we deal with the TE mode as it does not exclude the existence of other components of the H-field. The direction of the incident plane wave makes an arbitrary angle θ_i with the positive z-axis. The validity of the infinite-order simultaneous linear equations for the scattering coefficients of the TE scattering problem at oblique incidence and their associated electric and magnetic fields obtained in this paper can be deduced from the vertical polarization solution [11] by application of the principle of duality. In addition, the elementary function representations of *Schlömilch series* introduced by Twersky [1] are modified in such a way as to conform with the obliquely incident TE scattering solution presented here. Using the Sommerfeld integral representation of the Hankel function and the Poisson summation formula, *propagating and evanescent mode operators* associated with *Schlömilch series* have been derived. Upon employing these newly derived mode operators associated with oblique solution, the elementary function representations corresponding to Bessel and Neumann series have been acquired.

2. Formulation

A horizontally polarized plane electromagnetic wave obliquely incident upon the infinite grating of circular dielectric cylinders is indicated in figure 1. The dielectric cylinders of the infinite grating are placed perpendicularly to the xy-plane, separated by a distance of d. The infinite grating consists of an infinite number of parallel circular insulating dielectric cylinders of infinite length. The dielectric cylinders of the grating all have identical relative permittivity and permeability of ε_r and μ_r respectively. The radius of each individual dielectric cylinder in the array corresponding to the sth cylinder is denoted by a_s.

An arbitrarily polarized plane electromagnetic wave that is incident upon the infinite grating of insulating dielectric circular cylinders can be expressed as

$$\vec{E}^{\mathrm{inc}}(\vec{\rho}, t) = \mathrm{Re}\,\{\vec{E}_0 \mathrm{e}^{\mathrm{i}(\vec{k}_0\cdot\vec{\rho}-\omega t)}\} \tag{1}$$

where $\vec{E}_0$ is the constant complex electric field vector of arbitrary direction, $\vec{k}_0$ is a vector in the direction of propagation with magnitude k_0 that denotes the free-space wavenumber, $\vec{\rho}$ is the arbitrary position vector in the (x, y, z) Cartesian coordinate system, ω is the angular frequency of the arbitrarily polarized obliquely incident wave in radians and t stands for the time in seconds. The incident complex transverse electric field can be represented as

$$\vec{E}_h^{\mathrm{inc}} = \hat{h}_i E_{0h} \mathrm{e}^{\mathrm{i}\vec{k}_0\cdot\vec{\rho}} \tag{2}$$

where $\hat{h}_i$ is the horizontal polarization vector. $\vec{k}_0$ and $\vec{\rho}$ can be decomposed into their radial and z-components in the (r, ϕ, z) cylindrical coordinate system as

$$\vec{k}_0 = \vec{k}_r + \vec{k}_z \tag{3a}$$

$$\vec{\rho} = \vec{r} + \vec{z} \tag{3b}$$

$$\vec{r} = r(\hat{x}\cos\phi + \hat{y}\sin\phi) \tag{3c}$$

$$\vec{z} = z\hat{z} \tag{3d}$$

$$\vec{k}_r = -k_r(\hat{x}\cos\phi_i + \hat{y}\sin\phi_i) \tag{3e}$$

$$\vec{k}_z = -k_z\hat{z} \tag{3f}$$

where ϕ_i represents the angle of incidence in the xy-plane measured from the x-axis such that $\phi_i = \psi_i - \pi$,

$$k_r = k_0 \sin\theta_i \tag{3g}$$

$$k_z = k_0 \cos\theta_i \tag{3h}$$

where θ_i denotes the angle of incidence between $\vec{k}_0$ and the z-axis,

$$\vec{R}_s = \vec{r} - \vec{r}_s \tag{3i}$$

$$r = \sqrt{x^2 + y^2} \tag{3j}$$

$$\vec{r}_s = s\,\mathrm{d}\hat{y} \qquad \forall s \ni s \in Z \tag{3k}$$

$$\vec{R}_s = R_s(\hat{x}\cos\phi_s + \hat{y}\sin\phi_s) \tag{3l}$$

and Z is the set of all integers. We have obtained the incident transverse electric field in the cylindrical coordinate system of the sth cylinder [11] as

$$\vec{E}_h^{\mathrm{inc}}(R_s, \phi_s, z) = \hat{h}_i E_{0h} \mathrm{e}^{\mathrm{i}\vec{k}_r \cdot \vec{r}_s} \left\{ \sum_{n=-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}n\psi_i} J_n(k_r R_s) \mathrm{e}^{\mathrm{i}n(\phi_s + \pi/2)} \right\} \mathrm{e}^{-\mathrm{i}k_z z} \tag{4}$$

where ϕ_s denotes the angle between $\vec{r} - \vec{r}_s$ and the x-axis. The solution for the electric field intensity $\vec{E}(R_s, \phi_s, z)$ and the magnetic field intensity $\vec{H}(R_s, \phi_s, z)$ can be obtained for the internal and external regions of the cylinders separately in the coordinate system of any one of the cylinders. The entire external region that is outside all the insulating dielectric cylinders is referred to as *region I*, and the entire internal region that contains all the points inside all the insulating dielectric cylinders is referred to as *region II*. Hence, we adopt the following notation for the z-component of the magnetic field intensity, which is a function of R_s, ϕ_s, and z:

$$H_z(R_s, \phi_s, z) = H_z^{(\mathrm{I},h)}(R_s, \phi_s, z) \ni \{R_s \geqslant a_s \cap \vec{R}_s \text{ outside the cylinders}\} \tag{5a}$$

$$H_z(R_s, \phi_s, z) = H_z^{(\mathrm{II},h)}(R_s, \phi_s, z) \ni \{R_s \leqslant a_s \cap \vec{R}_s \text{ inside the cylinders}\}. \tag{5b}$$

Since the infinite grating consists of insulating dielectric circular cylinders of infinite length, the resultant fields must be periodic in the z-direction. Introducing this condition into (5), we can write

$$H_z^{(N,h)}(R_s, \phi_s, z) = \tilde{H}_z^{(N,h)}(R_s, \phi_s)\mathrm{e}^{\mathrm{i}k_z z} \qquad \forall N \ni N \in \{\mathrm{I}, \mathrm{II}\} \tag{6}$$

where k_z is given by (3h). Inserting (5) into the well known Maxwell equations of classical electromagnetism for time-harmonic fields, we have obtained the homogeneous Helmholtz equation associated with the z-component of the magnetic field intensity as

$$(\nabla^2 + k_N^2) H_z^{(N,h)}(R_s, \phi_s, z) = 0 \qquad \forall N \ni N \in \{\mathrm{I}, \mathrm{II}\} \tag{7}$$

where

$$k_{\mathrm{I}} = k_0 \qquad \{R_s \leqslant a_s\} \cap \{\vec{R}_s \text{ inside the cylinders}\} \tag{8a}$$

$$k_{\mathrm{II}} = k_0\sqrt{\varepsilon_r \mu_r} \qquad \{R_s \leqslant a_s\} \cap \{\vec{R}_s \text{ outside the cylinders}\}. \tag{8b}$$

We can express the Laplacian operator in the cylindrical coordinate system of the sth cylinder as

$$\nabla^2 = \nabla_t^2 + \frac{\partial^2}{\partial z^2} \tag{9a}$$

where

$$\nabla_t^2 = \frac{\partial^2}{\partial R_s^2} + \frac{1}{R_s}\frac{\partial}{\partial R_s} + \frac{1}{R_s^2}\frac{\partial^2}{\partial \phi_s^2}. \tag{9b}$$

Inserting (9a) in (7), we have the scalar Helmholtz equation associated with the z-components of the magnetic field intensities as

$$\left(\nabla_t^2 + \frac{\partial^2}{\partial z^2} + k_N^2\right)\tilde{H}_z^{(N,h)}(R_s,\phi_s)\mathrm{e}^{-\mathrm{i}k_z z} = 0 \qquad \forall N \ni N \in \{\mathrm{I}, \mathrm{II}\} \tag{10}$$

and

$$[\nabla_t^2 + (k_N^2 - k_z^2)]\tilde{H}_z^{(N,h)}(R_s,\phi_s) = 0 \qquad \forall N \ni N \in \{\mathrm{I}, \mathrm{II}\}. \tag{11}$$

This yields the equations for the z-component of the external magnetic field intensity as

$$(\nabla_t^2 + k_r^2)\tilde{H}_z^{(\mathrm{I},h)}(R_s,\phi_s) = 0 \ni \{R_s \geqslant a_s \cap \vec{R}_s \text{ outside the cylinders}\} \tag{12a}$$

and for the z-component of the internal magnetic field intensity as

$$(\nabla_t^2 + k_1^2)\tilde{H}_z^{(\mathrm{II},h)}(R_s,\phi_s) = 0 \ni \{R_s \leqslant a_s \cap \vec{R}_s \text{ inside the cylinders}\} \tag{12b}$$

where k_r is given by (3g) and k_1 is defined as

$$k_1 = k_0\sqrt{\varepsilon_r\mu_r - \cos^2\theta_i}. \tag{13}$$

3. Multiple-scattering representations for the obliquely incident transverse electric field

3.1. The z-components of the fields

The solution for the z-components of the magnetic field intensity in the external region can be written as

$$H_z^{(\mathrm{I},h)}(R_s,\phi_s,z) = H_z^{\mathrm{inc}}(R_s,\phi_s,z) + \sum_{j=-\infty}^{+\infty} H_z^j(R_j,\phi_j,z) \tag{14}$$

when there are cylinders whose axes are located at $\vec{r}_0, \vec{r}_1, \vec{r}_2$ etc. The first term in this expression corresponds to the z-component of the incident electric field intensity in the coordinate system of the sth cylinder that is located at $\vec{r}_s$. The second term $H_z^j(R_j,\phi_j,z)$ represents a cylindrical wave outgoing from the jth scatterer as $|\vec{r} - \vec{r}_j| \to \infty$. The z-component of the cylindrical wave scattered by the sth cylinder can be expressed in terms of Hankel functions of the first kind as

$$\tilde{H}_z^s(R_s,\phi_s) = \mathrm{e}^{\mathrm{i}\vec{k}_r\cdot\vec{r}_s}\sum_{n=-\infty}^{+\infty} C_{ns}H_n^{(1)}(k_r R_s)\mathrm{e}^{\mathrm{i}n(\phi_s+\pi/2)} \tag{15}$$

$$H_z^s(R_s,\phi_s,z) = \tilde{H}_z^s(R_s,\phi_s)\mathrm{e}^{-\mathrm{i}k_z z} \tag{16}$$

since the solution is performed in the coordinate system of the sth cylinder, that is the (R_s,ϕ_s,z) cylindrical coordinate system referred to the cylinder s. In the representation above, C_{ns} denote the undetermined scattering coefficients associated with the external magnetic field intensity referred to the sth cylinder for an obliquely incident transverse electric field.

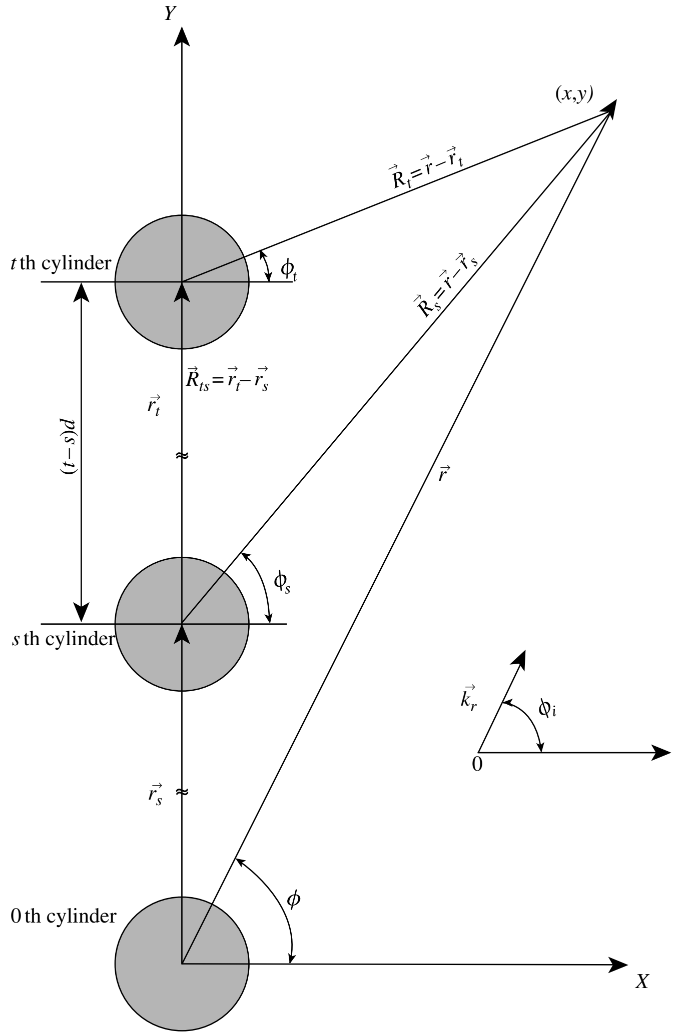

Figure 2. Descriptions of the variables in the TE scattering by the infinite grating.

In order to apply the boundary conditions on the surface of the sth cylinder $\vec{R}_s = \vec{a}_s$, it is necessary to express $\tilde{H}_z^t(R_t, \phi_t)$ for $t \neq s$ in terms of the cylindrical waves referred to the coordinate system of the sth cylinder. Referring to figure 2, we can obtain the z-component of the scattered magnetic field intensity by the tth cylinder in terms of the cylindrical waves referred to the coordinate system of the sth cylinder by applying the addition theorem for the cylindrical waves [2, 3, 11, 13] to the Hankel function $H_m^{(1)}(k_r R_t)$ of $\tilde{H}_z^t(R_t, \phi_t)$ as

$$\tilde{H}_z^t(R_s, \phi_s) = \mathrm{e}^{\mathrm{i}\vec{k}_r\cdot\vec{r}_t} \sum_{n=-\infty}^{+\infty} U_{nst} J_n(k_r R_s) \mathrm{e}^{\mathrm{i}n(\phi_s+\pi/2)} \tag{17}$$

where

$$U_{nst} = \sum_{m=-\infty}^{+\infty} C_{mt} H_{n-m}^{(1)}(k_r|\vec{r}_t - \vec{r}_s|) \mathrm{e}^{\mathrm{i}(m-n)(\phi_{st}-\frac{\pi}{2})} \tag{18}$$

and ϕ_{ts} is the angle made by $\vec{r}_t - \vec{r}_s$ with the x-axis. The z-component of the total magnetic field intensity in the external region can be expressed as

$$H_z^{(\mathrm{I},h)}(R_s,\phi_s,z) = H_z^{\mathrm{inc}}(R_s,\phi_s,z) + H_z^{s}(R_s,\phi_s,z) + \sum_{\substack{t=-\infty\\ t\neq s}}^{+\infty} H_z^{t}(R_t,\phi_t,z) \tag{19}$$

that is, the sum of the incident field and the scattered field from all cylinders. The z-component of the total magnetic field intensity in the external region of the cylinders can be obtained employing the expansion of a plane wave in terms of cylindrical waves referred to the coordinate system of the sth cylinder as

$$H_z^{(\mathrm{I},h)}(R_s,\phi_s,z) = \left\{ \mathrm{e}^{\mathrm{i}\vec{k}_r\cdot\vec{r}_s} \sum_{n=-\infty}^{+\infty} [(H_n^i + V_{ns})J_n(k_r R_s) + C_{ns} H_n^{(1)}(k_r R_s)]\mathrm{e}^{\mathrm{i}n(\phi_s+\pi/2)} \right\} \mathrm{e}^{-\mathrm{i}k_z z} \tag{20}$$

for $R_s \geqslant a_s$, where

$$H_n^i = \sin\theta_i H_{0v}\mathrm{e}^{-\mathrm{i}n\psi_i} \tag{21a}$$

$$H_{0v} = \eta_0 E_{0h} \tag{21b}$$

$$\eta_0 = \sqrt{\varepsilon_0/\mu_0}. \tag{21c}$$

The multiple-scattering terms associated with the magnetic field intensity in (20) are given as

$$V_{ns} = \sum_{\substack{t=-\infty\\ t\neq s}}^{+\infty} \mathrm{e}^{\mathrm{i}\vec{k}_r\cdot(\vec{r}_t-\vec{r}_s)} U_{nst} \tag{22a}$$

$$U_{nst} = \sum_{m=-\infty}^{+\infty} C_{mt} H_{n-m}^{(1)}(k_r|\vec{r}_t-\vec{r}_s|)\mathrm{e}^{\mathrm{i}(m-n)(\phi_{st}-\frac{\pi}{2})}. \tag{22b}$$

On the other hand, the internal magnetic field intensity can be expressed as

$$H_z^{(\mathrm{II},h)}(R_s,\phi_s,z) = \left\{ \mathrm{e}^{\mathrm{i}\vec{k}_r\cdot\vec{r}_s} \sum_{n=-\infty}^{+\infty} D_{ns} J_n(k_1 R_s)\mathrm{e}^{\mathrm{i}n(\phi_s+\pi/2)} \right\} \mathrm{e}^{-\mathrm{i}k_z z} \tag{23}$$

$R_s \leqslant a_s$, where k_1 is given by (13), and D_{ns} represent the undetermined scattering coefficients associated with the internal magnetic field intensity referred to the sth cylinder for an obliquely incident transverse electric field. In a similar fashion, we can write

$$E_z^{(\mathrm{I},h)}(R_s,\phi_s,z) = \left\{ \mathrm{e}^{\mathrm{i}\vec{k}_r\cdot\vec{r}_s} \sum_{n=-\infty}^{+\infty} [V_{ns}^E J_n(k_r R_s) + C_{ns}^E H_n^{(1)}(k_r R_s)]\mathrm{e}^{\mathrm{i}n(\phi_s+\pi/2)} \right\} \mathrm{e}^{-\mathrm{i}k_z z} \tag{24}$$

$R_s \geqslant a_s$, for the external electric field intensity, and the multiple-scattering terms associated with this field are given as

$$V_{ns}^E = \sum_{\substack{t=-\infty\\ t\neq s}}^{+\infty} \mathrm{e}^{\mathrm{i}\vec{k}_r\cdot(\vec{r}_t-\vec{r}_s)} U_{nst}^E \tag{25a}$$

$$U_{nst}^E = \sum_{m=-\infty}^{+\infty} C_{mt}^E H_{n-m}^{(1)}(k_r|\vec{r}_t-\vec{r}_s|)\mathrm{e}^{\mathrm{i}(m-n)(\phi_{st}-\frac{\pi}{2})}. \tag{25b}$$

On the other hand, we can write

$$E_z^{(\mathrm{II},h)}(R_s,\phi_s,z) = \left\{ \mathrm{e}^{\mathrm{i}\vec{k}_r\cdot\vec{r}_s} \sum_{n=-\infty}^{+\infty} D_{ns}^E J_n(k_1 R_s)\mathrm{e}^{\mathrm{i}n(\phi_s+\pi/2)} \right\} \mathrm{e}^{-\mathrm{i}k_z z} \tag{26}$$

for $R_s \leqslant a_s$, for the internal electric field intensity, respectively. C_{ns}^E and D_{ns}^E represent the undetermined scattering coefficients referring to the external and internal regions, $\forall n \ni n \in Z$ and $\forall s \ni s \in Z$ corresponding to $E_z^{(M,h)} \ni M \in \{\mathrm{I},\mathrm{II}\}$, z-components of the electric field intensities.

3.2. Angular components of the electric and magnetic field intensities

Angular components of the electric and magnetic field intensities associated with the external fields can be derived from Maxwell's equations in terms of the z-components of the electric and magnetic field intensities as

$$H_{\phi_s}^{(\mathrm{I},h)}(R_s,\phi_s,z) = \frac{1}{k_r^2}\frac{1}{R_s}\frac{\partial}{\partial z}\left(\frac{\partial H_z^{(\mathrm{I},h)}}{\partial \phi_s}\right) - \frac{\omega\varepsilon_0}{\mathrm{i}k_r^2}\left(\frac{\partial E_z^{(\mathrm{I},h)}}{\partial R_s}\right) \tag{27}$$

$$\begin{aligned} H_{\phi_s}^{(\mathrm{I},h)}(R_s,\phi_s,z) = \mathrm{e}^{-\mathrm{i}k_z z}\mathrm{e}^{\mathrm{i}\vec{k}_r\cdot\vec{r}_s}\sum_{n=-\infty}^{+\infty}\Bigg\{&\frac{nk_z}{R_s k_r^2}[(H_n^i + V_{ns})J_n(k_r R_s) + C_{ns}H_n^{(1)}(k_r R_s)] \\ &-\frac{\omega\varepsilon_0}{\mathrm{i}k_r}[V_{ns}^E \dot{J}_n(k_r R_s) + C_{ns}^E \dot{H}_n^{(1)}(k_r R_s)]\Bigg\}\mathrm{e}^{\mathrm{i}n(\phi_s+\pi/2)} \end{aligned} \tag{28}$$

and

$$E_{\phi_s}^{(\mathrm{I},h)}(R_s,\phi_s,z) = \frac{1}{k_r^2}\frac{1}{R_s}\frac{\partial}{\partial z}\left(\frac{\partial E_z^{(\mathrm{I},h)}}{\partial \phi_s}\right) + \frac{\omega\mu_0}{\mathrm{i}k_r^2}\left(\frac{\partial H_z^{(\mathrm{I},h)}}{\partial R_s}\right) \tag{29}$$

$$\begin{aligned} E_{\phi_s}^{(\mathrm{I},h)}(R_s,\phi_s,z) = \mathrm{e}^{-\mathrm{i}k_z z}\mathrm{e}^{\mathrm{i}\vec{k}_r\cdot\vec{r}_s}\sum_{n=-\infty}^{+\infty}\Bigg\{&\frac{nk_z}{R_s k_r^2}[V_{ns}^E J_n(k_r R_s) + C_{ns}^E H_n^{(1)}(k_r R_s)] \\ &+\frac{\omega\mu_0}{\mathrm{i}k_r}[(H_n^i + V_{ns})\dot{J}_n(k_r R_s) + C_{ns}\dot{H}_n^{(1)}(k_r R_s)]\Bigg\}\mathrm{e}^{\mathrm{i}n(\phi_s+\pi/2)} \end{aligned} \tag{30}$$

in the external region, $R_s \geqslant a_s$. In the above, the $\dot{J}_n$ and $\dot{H}_n^{(1)}$ are defined as $\dot{J}_n(\zeta) \equiv (\mathrm{d}/\mathrm{d}\zeta)J_n(\zeta)$ and $\dot{H}_n^{(1)}(\zeta) \equiv (\mathrm{d}/\mathrm{d}\zeta)H_n^{(1)}(\zeta)$, which are the first derivatives of the Bessel and Hankel functions of first kind and of order n with respect to their arguments. Similarly, in the internal region, we can calculate the magnetic field intensity using the expression

$$H_{\phi_s}^{(\mathrm{II},h)}(R_s,\phi_s,z) = \frac{1}{k_1^2}\frac{1}{R_s}\frac{\partial}{\partial z}\left(\frac{\partial H_z^{(\mathrm{II},h)}}{\partial \phi_s}\right) - \frac{\omega\varepsilon_1}{\mathrm{i}k_1^2}\left(\frac{\partial E_z^{(\mathrm{II},h)}}{\partial R_s}\right) \tag{31}$$

that yields

$$H_{\phi_s}^{(\mathrm{II},h)}(R_s,\phi_s,z) = \mathrm{e}^{\mathrm{i}\vec{k}_r\cdot\vec{r}_s}\left\{\sum_{n=-\infty}^{+\infty}\left[\frac{nk_z}{R_s k_1^2}D_{ns}J_n(k_1 R_s) - \frac{\omega\varepsilon_1}{\mathrm{i}k_1}D_{ns}^E \dot{J}_n(k_1 R_s)\right]\mathrm{e}^{\mathrm{i}n(\phi_s+\frac{\pi}{2})}\right\}\mathrm{e}^{-\mathrm{i}k_z z}. \tag{32}$$

The angular component of the electric field intensity can be expressed in terms of the z-components of the electric and magnetic field intensities in the internal region as

$$E_{\phi_s}^{(\mathrm{II},h)}(R_s,\phi_s,z) = \frac{1}{k_1^2}\frac{1}{R_s}\frac{\partial}{\partial z}\left(\frac{\partial E_z^{(\mathrm{II},h)}}{\partial \phi_s}\right) + \frac{\omega\mu_1}{\mathrm{i}k_1^2}\left(\frac{\partial H_z^{(\mathrm{II},h)}}{\partial R_s}\right) \tag{33}$$

that provides the desired angular component of the electric field intensity in the internal region as

$$E_{\phi_s}^{(\mathrm{II},h)}(R_s,\phi_s,z) = \mathrm{e}^{\mathrm{i}\vec{k}_r\cdot\vec{r}_s}\left\{\sum_{n=-\infty}^{+\infty}\left[\frac{nk_z}{R_s k_1^2}D_{ns}^E J_n(k_1 R_s) + \frac{\omega\mu_1}{\mathrm{i}k_1}D_{ns}\dot{J}_n(k_1 R_s)\right]\mathrm{e}^{\mathrm{i}n(\phi_s+\pi/2)}\right\}\mathrm{e}^{-\mathrm{i}k_z z} \tag{34}$$

for $R_s \leqslant a_s$.

3.3. Radial components of the electric and magnetic field intensities

Using the Maxwell equations, we can express the radial component of the magnetic field intensity in terms of the angular and z-components of the external electric field intensities as

$$H_{R_s}^{(\mathrm{I},h)}(R_s,\phi_s,z)=\frac{1}{\mathrm{i}\omega\mu_0}\left[\frac{1}{R_s}\frac{\partial E_z^{(\mathrm{I},h)}}{\partial\phi_s}+\mathrm{i}k_zE_{\phi_s}^{(\mathrm{I},h)}\right] \tag{35}$$

which yields the radial component of the external magnetic field intensity in terms of the Bessel and Hankel functions and some undetermined coefficients as

$$\begin{aligned}H_{R_s}^{(\mathrm{I},h)}(R_s,\phi_s,z)=\mathrm{e}^{\mathrm{i}\vec{k}_r\cdot\vec{r}_s}\Bigg\{\sum_{n=-\infty}^{+\infty}\Bigg\{\frac{n}{\omega\mu_0}\left[1+\left(\frac{k_z}{k_r}\right)^2\right]\frac{1}{R_s}[V_{ns}^EJ_n(k_rR_s)+C_{ns}^EH_n^{(1)}(k_rR_s)]\\-\mathrm{i}\left(\frac{k_z}{k_r}\right)[(H_n^i+V_{ns})\dot{J}_n(k_rR_s)+C_{ns}\dot{H}_n^{(1)}(k_rR_s)]\Bigg\}\mathrm{e}^{\mathrm{i}n(\phi_s+\pi/2)}\Bigg\}\mathrm{e}^{-\mathrm{i}k_zz}.\end{aligned} \tag{36}$$

Similarly, the radial component of the electric field intensity in the external region of the infinite grating can be expressed using the angular and z-components of the magnetic field intensities in the external region as

$$E_{R_s}^{(\mathrm{I},h)}(R_s,\phi_s,z)=\frac{-1}{\mathrm{i}\omega\varepsilon_0}\left[\frac{1}{R_s}\frac{\partial H_z^{(\mathrm{I},h)}}{\partial\phi_s}+\mathrm{i}k_zH_{\phi_s}^{(\mathrm{I},h)}\right] \tag{37}$$

which in turn yields the radial component of the electric field intensity in the external region in terms of the Bessel and Hankel functions and some undetermined coefficients as

$$\begin{aligned}E_{R_s}^{(\mathrm{I},h)}(R_s,\phi_s,z)=\mathrm{e}^{\mathrm{i}\vec{k}_r\cdot\vec{r}_s}\Bigg\{\sum_{n=-\infty}^{+\infty}\Bigg\{-\frac{n}{\omega\varepsilon_0}\left[1+\left(\frac{k_z}{k_r}\right)^2\right]\frac{1}{R_s}[(H_n^i+V_{ns})J_n(k_rR_s)\\+C_{ns}H_n^{(1)}(k_rR_s)]-\mathrm{i}\left(\frac{k_z}{k_r}\right)[V_{ns}^E\dot{J}_n(k_rR_s)+C_{ns}^E\dot{H}_n^{(1)}(k_rR_s)]\Bigg\}\mathrm{e}^{\mathrm{i}n(\phi_s+\pi/2)}\Bigg\}\mathrm{e}^{-\mathrm{i}k_zz}.\end{aligned} \tag{38}$$

For the internal region of the grating, we can express the radial component of the magnetic field intensity in terms of the angular and z-components of the electric field intensities as

$$H_{R_s}^{(\mathrm{II},h)}(R_s,\phi_s,z)=\frac{1}{\mathrm{i}\omega\mu_1}\left[\frac{1}{R_s}\frac{\partial E_z^{(\mathrm{II},h)}}{\partial\phi_s}+\mathrm{i}k_zE_{\phi_s}^{(\mathrm{II},h)}\right] \tag{39}$$

which yields the radial component of the magnetic field intensity in the internal region of the grating in terms of the Bessel and Hankel functions and some undetermined coefficients as

$$\begin{aligned}H_{R_s}^{(\mathrm{II},h)}(R_s,\phi_s,z)=\mathrm{e}^{\mathrm{i}\vec{k}_r\cdot\vec{r}_s}\Bigg\{\sum_{n=-\infty}^{+\infty}\frac{n}{\omega\mu_1}\left[1+\left(\frac{k_z}{k_1}\right)^2\right]\frac{1}{R_s}D_{ns}^EJ_n(k_1R_s)\\-\mathrm{i}\left(\frac{k_z}{k_1}\right)D_{ns}\dot{J}_n(k_1R_s)\Bigg\}\mathrm{e}^{\mathrm{i}n(\phi_s+\pi/2)}\Bigg\}\mathrm{e}^{-\mathrm{i}k_zz}.\end{aligned} \tag{40}$$

Similarly, the radial component of the electric field intensity in the internal region of the infinite grating can be expressed using the angular and z-components of the magnetic field intensities as

$$E_{R_s}^{(\mathrm{II},h)}(R_s,\phi_s,z)=\frac{-1}{\mathrm{i}\omega\varepsilon_1}\left[\frac{1}{R_s}\frac{\partial H_z^{(\mathrm{II},h)}}{\partial\phi_s}+\mathrm{i}k_zH_{\phi_s}^{(\mathrm{II},h)}\right] \tag{41}$$

which in turn yields the radial component of the electric field intensity in the internal region of the infinite grating in terms of the Bessel and Hankel functions and some undetermined coefficients as

$$E_{R_s}^{\mathrm{II},h}(R_s,\phi_s,z)=\mathrm{e}^{\mathrm{i}\vec{k_r}\cdot\vec{r_s}}\left\{\sum_{n=-\infty}^{+\infty}\left\{-\frac{n}{\omega\varepsilon_1}\left[1+\left(\frac{k_z}{k_1}\right)^2\right]\frac{1}{R_s}D_{ns}J_n(k_1R_s)\right.\right.$$
$$\left.\left.-\mathrm{i}\left(\frac{k_z}{k_1}\right)D_{ns}^E\dot{J}_n(k_1R_s)\right\}\mathrm{e}^{\mathrm{i}n(\phi_s+\pi/2)}\right\}\mathrm{e}^{-\mathrm{i}k_zz}. \tag{42}$$

3.4. Application of boundary conditions

Using the continuity of the z-components of the field intensities at the surface of each cylinder, we obtained

$$(H_n^i+V_{ns})J_n(k_ra_s)+C_{ns}H_n^{(1)}(k_ra_s)=D_{ns}J_n(k_1a_s) \tag{43}$$

and

$$V_{ns}^EJ_n(k_ra_s)+C_{ns}^EH_n^{(1)}(k_ra_s)=D_{ns}^EJ_n(k_1a_s). \tag{44}$$

Similarly, employing the continuity of the angular components of the field intensities at the surface of each cylinder, we have

$$\frac{nk_z}{a_sk_r^2}[(H_n^i+V_{ns})J_n(k_ra_s)+C_{ns}H_n^{(1)}(k_ra_s)]-\frac{\omega\varepsilon_0}{\mathrm{i}k_r}[V_{ns}^E\dot{J}_n(k_ra_s)+C_{ns}^E\dot{H}_n^{(1)}(k_ra_s)]$$
$$=\frac{nk_z}{a_sk_1^2}D_{ns}J_n(k_1a_s)-\frac{\omega\varepsilon_1}{\mathrm{i}k_1}D_{ns}^E\dot{J}_n(k_1a_s) \tag{45}$$

and

$$\frac{nk_z}{a_sk_r^2}[V_{ns}^EJ_n(k_ra_s)+C_{ns}^EH_n^{(1)}(k_ra_s)]+\frac{\omega\mu_0}{\mathrm{i}k_r}[(H_n^i+V_{ns})\dot{J}_n(k_ra_s)+C_{ns}\dot{H}_n^{(1)}(k_ra_s)]$$
$$=\frac{nk_z}{a_sk_1^2}D_{ns}^EJ_n(k_1a_s)+\frac{\omega\mu_1}{\mathrm{i}k_1}D_{ns}\dot{J}_n(k_1a_s). \tag{46}$$

Under the assumption that all the cylinders have the same radius, i.e. $a_s=a$ for all s, the coefficients of the internal magnetic and electric field intensities inside the infinite grating can be expressed in terms of the scattering coefficients of the external field intensities as

$$D_n=\frac{(H_n^i+V_n)J_n(k_ra)+C_nH_n^{(1)}(k_ra)}{J_n(k_1a)} \tag{47}$$

for the magnetic field intensity, and

$$D_n^E=\frac{V_n^EJ_n(k_ra)+C_n^EH_n^{(1)}(k_ra)}{J_n(k_1a)} \tag{48}$$

for the electric field intensity. The scattering coefficients of the external magnetic and electric fields are expressed in terms of the following two infinite sets of equations:

$$\frac{nk_z}{a}\left(\frac{1}{k_r^2}-\frac{1}{k_1^2}\right)[(H_n^i+V_n)J_n(k_ra)+C_nH_n^{(1)}(k_ra)]-\frac{\omega\varepsilon_0}{\mathrm{i}k_r}[V_n^E\dot{J}_n(k_ra)+C_n^E\dot{H}_n^{(1)}(k_ra)]$$
$$=-\frac{\omega\varepsilon_1}{\mathrm{i}k_1}[V_n^EJ_n(k_ra)+C_n^EH_n^{(1)}(k_ra)]\frac{\dot{J}_n(k_1a)}{J_n(k_1a)} \tag{49}$$

and

$$\frac{nk_z}{a}\left(\frac{1}{k_r^2}-\frac{1}{k_1^2}\right)[V_n^EJ_n(k_ra)+C_n^EH_n^{(1)}(k_ra)]+\frac{\omega\mu_0}{\mathrm{i}k_r}[(H_n^i+V_n)\dot{J}_n(k_ra)+C_n\dot{H}_n^{(1)}(k_ra)]$$
$$=\frac{\omega\mu_1}{\mathrm{i}k_1}[(H_n^i+V_n)J_n(k_ra)+C_nH_n^{(1)}(k_ra)]\frac{\dot{J}_n(k_1a)}{J_n(k_1a)}. \tag{50}$$

The equations for the scattering coefficients of the infinite grating of insulating dielectric circular cylinders can be further simplified by defining a constant K_n as

$$K_n = \frac{nk_z}{a}\left(\frac{1}{k_1^2} - \frac{1}{k_r^2}\right) \tag{51}$$

$\forall n \ni n \in Z$, and two sets of constants $\alpha_n(\zeta_r)$ and $\beta_n(\zeta_r)$, in which $\zeta_r \in \{\varepsilon_r, \mu_r\}$ represents the dielectric constant and the relative permeability of the dielectric cylinders respectively, as

$$\alpha_n(\zeta_r) = \left[\frac{\dot{J}_n(k_r a)}{k_r} - \left(\frac{\zeta_r}{k_1}\right)\frac{J_n(k_r a)\dot{J}_n(k_1 a)}{J_n(k_1 a)}\right] \tag{52a}$$

$$\beta_n(\zeta_r) = \left[\frac{\dot{H}_n^{(1)}(k_r a)}{k_r} - \left(\frac{\zeta_r}{k_1}\right)\frac{H_n^{(1)}(k_r a)\dot{J}_n(k_1 a)}{J_n(k_1 a)}\right] \tag{52b}$$

$\forall n \ni n \in Z$. Introducing these definitions into (49) and (50), the latter equations can be written in simplified form as

$$\frac{K_n H_n^{(1)}(k_r a)}{\mathrm{i}\omega\varepsilon_0\beta_n(\varepsilon_r)}\left[C_n + \frac{J_n(k_r a)}{H_n^{(1)}(k_r a)}(H_n^i + V_n)\right] = \left[C_n^E + \frac{\alpha_n(\varepsilon_r)}{\beta_n(\varepsilon_r)}V_n^E\right] \qquad \forall n \ni n \in Z \tag{53a}$$

$$\frac{K_n H_n^{(1)}(k_r a)}{\mathrm{i}\omega\mu_0\beta_n(\mu_r)}\left[C_n^E + \frac{J_n(k_r a)}{H_n^{(1)}(k_r a)}V_n^E\right] = -\left[C_n + \frac{\alpha_n(\mu_r)}{\beta_n(\mu_r)}(H_n^i + V_n)\right] \qquad \forall n \ni n \in Z. \tag{53b}$$

3.5. Evaluation of the multiple-scattering terms V_n and V_n^E

The multiple-scattering effects associated with the magnetic field intensities were given in (22). Inserting (22*b*) into (22*a*), and assuming that all the cylinders are identical, we can write the multiple-interaction terms associated with magnetic field intensity as

$$V_n = \sum_{\substack{t=-\infty \\ t\neq s}}^{+\infty} \mathrm{e}^{\mathrm{i}\vec{k}_r\cdot(\vec{r}_t-\vec{r}_s)}\left\{\sum_{m=-\infty}^{+\infty} C_m H_{n-m}^{(1)}(k_r|\vec{r}_t - \vec{r}_s|)\mathrm{e}^{\mathrm{i}(m-n)(\phi_{st}-\frac{\pi}{2})}\right\}. \tag{54a}$$

Interchanging the order of summation as in Twersky's treatment [3] for the non-oblique case, we have

$$V_n = \sum_{m=-\infty}^{+\infty} C_m\left\{\sum_{\substack{t=-\infty \\ t\neq s}}^{+\infty} \mathrm{e}^{\mathrm{i}\vec{k}_r\cdot(\vec{r}_t-\vec{r}_s)} H_{n-m}^{(1)}(k_r|\vec{r}_t - \vec{r}_s|)\mathrm{e}^{\mathrm{i}(m-n)(\phi_{st}-\frac{\pi}{2})}\right\}. \tag{54b}$$

We recognize the term under the summation as a *Schlömilch series* [3]

$$\mathcal{J}_{n-m}(k_r d) := \sum_{\substack{t=-\infty \\ t\neq s}}^{+\infty} \mathrm{e}^{\mathrm{i}\vec{k}_r\cdot(\vec{r}_t-\vec{r}_s)} H_{n-m}^{(1)}(k_r|\vec{r}_t - \vec{r}_s|)\mathrm{e}^{\mathrm{i}(m-n)(\phi_{st}-\frac{\pi}{2})} \tag{55a}$$

where $H_n^{(1)} = J_n + \mathrm{i}N_n$ represents the Hankel function of the first kind and J_n and N_n the *Bessel* and *Neumann* functions. The *Schlömilch series* are convergent provided that $k_r d(1\pm\sin\psi_i)/2\pi$ is not equal to integers [1,3,13]. Introducing the definition of the *Schlömilch series* as in [1] as

$$\mathcal{J}_{n-m}(k_r d) = \sum_{s=1}^{\infty} H_{n-m}^{(1)}(sk_r d)[\mathrm{e}^{-\mathrm{i}sk_r d\sin\psi_i} + (-1)^{n-m}\mathrm{e}^{\mathrm{i}sk_r d\sin\psi_i}] \tag{55b}$$

the multiple-scattering effects associated with magnetic field intensity can be expressed in terms of *Schlömilch series* as

$$V_n = \sum_{m=-\infty}^{+\infty} C_m \mathcal{J}_{n-m}(k_r d). \tag{56}$$

In order to evaluate the *multiple-scattering effects* V_n^E' associated with the electric field intensities, we insert (25*b*) into (25*a*) and assume that all the cylinders are identical, thus we have

$$V_n^E = \sum_{\substack{t=-\infty \\ t\neq s}}^{+\infty} \mathrm{e}^{\mathrm{i}\vec{k}_r\cdot(\vec{r}_t-\vec{r}_s)} \left\{ \sum_{m=-\infty}^{+\infty} C_m^E H_{n-m}^{(1)}(k_r|\vec{r}_t-\vec{r}_s|)\mathrm{e}^{\mathrm{i}(m-n)(\phi_{st}-\frac{\pi}{2})} \right\}. \tag{57}$$

Interchanging the order of summation in (57) as in Twersky's treatment [3], we obtained

$$V_n^E = \sum_{m=-\infty}^{+\infty} C_n^E \mathcal{J}_{n-m}(k_r d) \tag{58}$$

where $\mathcal{J}_{n-m}(k_r d)$ is the *Schlömilch series* and is given by (55*b*).

3.6. Exact equations for the scattering coefficients for the obliquely incident transverse electric field

The exact form of the equations for the scattering coefficients of the transverse electric multiple scattering of the infinite grating at oblique incidence can then be expressed in terms of the well known *Schlömilch series* as

$$\begin{aligned} &\frac{K_n H_n^{(1)}(k_r a)}{\mathrm{i}\omega\varepsilon_0\beta_n(\varepsilon_r)}\left\{C_n + \frac{J_n(k_r a)}{H_n^{(1)}(k_r a)}\left[H_n^i + \sum_{m=-\infty}^{+\infty} C_m \mathcal{J}_{n-m}(k_r d)\right]\right\} \\ &\qquad = \left[C_n^E + \frac{\alpha_n(\varepsilon_r)}{\beta_n(\varepsilon_r)}\sum_{m=-\infty}^{+\infty} C_m^E \mathcal{J}_{n-m}(k_r d)\right] \qquad \forall n \ni n \in Z \end{aligned} \tag{59a}$$

$$\begin{aligned} &\frac{K_n H_n^{(1)}(k_r a)}{\mathrm{i}\omega\mu_0\beta_n(\mu_r)}\left[C_n^E + \frac{J_n(k_r a)}{H_n^{(1)}(k_r a)}\sum_{m=-\infty}^{+\infty} C_m^E \mathcal{J}_{n-m}(k_r d)\right] \\ &\qquad = -\left\{C_n + \frac{\alpha_n(\mu_r)}{\beta_n(\mu_r)}\left[H_n^i + \sum_{m=-\infty}^{+\infty} C_m \mathcal{J}_{n-m}(k_r d)\right]\right\} \qquad \forall n \ni n \in Z. \end{aligned} \tag{59b}$$

In the above, we have obtained the scattering coefficients for the electric and magnetic fields for an obliquely incident transverse electric plane wave in terms of two systems of simultaneous linear equations of infinite order in coupled form.

4. Elementary function representations of the multiple-scattering terms for the transverse electric scattering at oblique incidence

4.1. Schlömilch series

In order to be able to evaluate the multiple-scattering terms analytically, we shall refer to Twersky's elementary function representations of *Schlömilch series*. This section demonstrates the application of these elementary function representations for the obliquely incident waves. In general, the multiple-scattering representation given in (55*b*) is too slowly convergent when the distance between the cylinders of the grating is much smaller than the wavelength of the radiation, i.e. $k_r d \ll 1$. Therefore, an alternative representation of the *Schlömilch series* will be presented in terms of elementary functions. For this purpose, we first put (55*b*) into a more compact form as

$$\mathcal{J}_{n-m}(k_r d) = \sum_{s=1}^{\infty}[H_{n-m}^{(1)}(sk_r d)\mathrm{e}^{-\mathrm{i}sk_r d \sin\psi_i} + (-1)^{n-m} H_{n-m}^{(1)}(sk_r d)\mathrm{e}^{\mathrm{i}sk_r d \sin\psi_i}] \tag{60a}$$

$$\mathcal{J}_{n-m}(k_r d) = \sum_{s=1}^{\infty} H_{n-m}^{(1)}(sk_r d)\mathrm{e}^{-\mathrm{i}sk_r d \sin\psi_i} + \sum_{s=1}^{\infty}(-1)^{n-m} H_{n-m}^{(1)}(sk_r d)\mathrm{e}^{\mathrm{i}sk_r d \sin\psi_i}. \tag{60b}$$

In the above, employing

$$H_{n-m}^{(1)}(-sk_r d) \equiv (-1)^{n-m} H_{n-m}^{(1)}(sk_r d) \tag{61}$$

we have obtained

$$\mathcal{J}_{n-m}(k_r d) = \sum_{\substack{s=-\infty \\ s\neq 0}}^{\infty} H_{n-m}^{(1)}(sk_r d)\mathrm{e}^{-\mathrm{i}sk_r d \sin\psi_i}. \tag{62}$$

In order to reduce $\mathcal{J}_{n-m}(k_r d)$ in (62) to its elementary function representation, which we shall denote by $\mathcal{H}$, we note that the sum over s in (62) is essentially the limit of the sum of $h_{n-m}(k_r r)$ for $y = x \to 0$, i.e.

$$\mathcal{J}_{n-m}(k_r d) = \lim_{\substack{r\to 0 \\ (y=x\to 0)}} h_{n-m}(k_r r) := \mathcal{H}_{n-m}(k_r d) \tag{63}$$

where r is given by $(3j)$ and $h_{n-m}(k_r r)$ are given for $x > 0$ as

$$h_{n-m}(k_r r) = \sum_{\substack{s=-\infty \\ s\neq 0}}^{\infty} \mathrm{e}^{\mathrm{i}sk_r d \sin\psi_i} H_{n-m}^{(1)}(k_r R_s)\mathrm{e}^{\mathrm{i}(\frac{\pi}{2}-\phi_s)(n-m)} \tag{64}$$

or, more explicitly, we can write

$$h_{n-m}(k_r r) = -H_{n-m}^{(1)}(k_r R_0)\mathrm{i}^{n-m}\mathrm{e}^{-\mathrm{i}(n-m)\phi_0} + \sum_{s=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}sk_r d \sin\psi_i} H_{n-m}^{(1)}(k_r R_s)\mathrm{i}^{n-m}\mathrm{e}^{-\mathrm{i}(n-m)\phi_s}. \tag{65}$$

In the derivation above, we have used the expressions pertaining to figure 2 as

$$R_s = \sqrt{x^2 + (y - sd)^2} \qquad \forall s \ni s \in Z \tag{66a}$$

$$\lim_{r\to 0} k_r R_s = -sk_r d \tag{66b}$$

$$R_0 \equiv \sqrt{x^2 + y^2} = r \tag{66c}$$

and

$$\phi_s = \tan^{-1}\left(\frac{y - sd}{x}\right) \qquad \forall s \ni s \in Z. \tag{67}$$

4.2. Derivation of Schlömilch series in terms of operators

Defining

$$\vec{k_\phi} = k_r(\hat{x}\cos\phi + \hat{y}\sin\phi) \tag{68a}$$

$$\vec{R_s} = R_s(\hat{x}\cos\phi_s + \hat{y}\sin\phi_s) \tag{68b}$$

$$\vec{R_s} = x\hat{x} + (y - sd)\hat{y} \tag{68c}$$

$$\vec{k_\phi}\cdot\vec{R_s} = k_r R_s \cos(\phi - \phi_s) \tag{69a}$$

$$\vec{k_\phi}\cdot\vec{R_s}(-\phi_s) = k_r R_s \cos(\phi + \phi_s) \tag{69b}$$

and using the *Sommerfeld integral representation* of the Hankel function and exploiting the full range available for the limits of the path in the complex domain, we can write for $x > 0$

$$H^{(1)}_{n-m}(k_r R_s)\mathrm{i}^{n-m}\mathrm{e}^{-\mathrm{i}(n-m)\phi_s} = \frac{1}{\pi}\int_{-\frac{\pi}{2}+\mathrm{i}\infty}^{\frac{\pi}{2}-\mathrm{i}\infty} \mathrm{e}^{\mathrm{i}\vec{k}_\phi\cdot\vec{R}_s(-\phi_s)+\mathrm{i}(n-m)\phi}\,\mathrm{d}\phi \tag{70a}$$

$$H^{(1)}_{n-m}(k_r R_s)\mathrm{i}^{n-m}\mathrm{e}^{-\mathrm{i}(n-m)\phi_s} = \frac{1}{\pi}\int_{-\frac{\pi}{2}+\mathrm{i}\infty}^{\frac{\pi}{2}-\mathrm{i}\infty} \mathrm{e}^{\mathrm{i}k_r R_s\cos(\phi+\phi_s)+\mathrm{i}(n-m)\phi}\,\mathrm{d}\phi \tag{70b}$$

$$H^{(1)}_{n-m}(k_r R_s)\mathrm{i}^{n-m}\mathrm{e}^{-\mathrm{i}(n-m)\phi_s} = \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}k_r\left[x\sqrt{1-p^2}-(y-sd)p\right]}\frac{\mathrm{e}^{\mathrm{i}(n-m)\sin^{-1}p}}{\sqrt{1-p^2}}\,\mathrm{d}p \tag{71}$$

which holds for all $|\phi_s| < \frac{\pi}{2}$, i.e. $x > 0$. Utilizing (71) in the further evaluation of the second term in (65), we can write

$$\begin{aligned}
&\sum_{s=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}sk_r d\sin\psi_i}H^{(1)}_{n-m}(k_r R_s)i^{n-m}\mathrm{e}^{-\mathrm{i}(n-m)\phi_s}\\
&\qquad= \sum_{s=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}sk_r d\sin\psi_i}\left\{\frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}k_r\left(x\sqrt{1-p^2}-(y-sd)p\right)}\left[\frac{\mathrm{e}^{\mathrm{i}(n-m)\sin^{-1}p}}{\sqrt{1-p^2}}\right]\mathrm{d}p\right\}\\
&\qquad= \left\{\frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}k_r\left(x\sqrt{1-p^2}-yp\right)}\left[\frac{\mathrm{e}^{\mathrm{i}(n-m)\sin^{-1}p}}{\sqrt{1-p^2}}\right]\mathrm{d}p\right\}\sum_{s=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}sk_r d(p+\sin\psi_i)}.
\end{aligned} \tag{72}$$

We now introduce the *Poisson summation formula* into (72) as

$$\sum_{s=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}sk_r d(p+\sin\psi_i)} = \frac{2\pi}{k_r d}\sum_{s=-\infty}^{\infty}\delta(p+\sin\phi_s) \tag{73}$$

where $\sin\phi_s$ is defined as

$$\sin\phi_s := \sin\psi_i + s\frac{2\pi}{k_r d}. \tag{74}$$

The angles ϕ_s are the usual *diffraction angles* of the grating, and equation (74), that provides these discrete angles, is called the *grating equation*, which appears in the single-scattering model as well. *Propagating modes* are determined by $|\sin\phi_s| < 1$, and they correspond to $|s| \leqslant s_\pm$, where s_+ and s_- are the closest integers to σ_+ and σ_- for which $|\sin\phi_s| < 1$ should be satisfied, i.e. $s_\pm < \sigma_\pm$, such that

$$\sigma_\pm = (1 \mp \sin\phi_i)\left(\frac{k_r d}{2\pi}\right). \tag{75}$$

Evanescent modes are determined by $|\sin\phi_s| > 1$, and they correspond to integer values of s such that $|s| \geqslant s_\pm + 1$; we have $\pm\sin\phi_s^\pm > 1$, and $\phi_s^\pm$ are determined by $\phi_s^\pm = \pm\frac{\pi}{2} \mp \mathrm{i}|\eta_s^\pm|$. For this case the *grating equation* takes the form

$$\cosh|\eta_s^\pm| = \pm\left[\sin\phi_i + s\left(\frac{2\pi}{k_r d}\right)\right] > 1 \qquad \forall s \ni \{s\in Z|\pm s \geqslant s_\pm + 1\}. \tag{76}$$

We have evaluated (72) as

$$\sum_{s=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}sk_r d\sin\psi_i}H^{(1)}_{n-m}(k_r R_s)\mathrm{i}^{n-m}\mathrm{e}^{-\mathrm{i}(n-m)\phi_s} = 2\sum_{s=-\infty}^{\infty}(L_s\mathrm{e}^{-\mathrm{i}(n-m)\phi_s})\mathrm{e}^{\mathrm{i}\vec{k}_s\cdot\vec{r}} \tag{77}$$

where L_s, $\vec{k}_s$ and $\vec{r}$ are defined as

$$L_s := \frac{1}{k_r d\cos\phi_s} \tag{78a}$$

$$\vec{k}_s := k_r(\hat{x}\cos\phi_s + \hat{y}\sin\phi_s) \tag{78b}$$

$$\vec{r} := x\hat{x} + y\vec{y}. \tag{78c}$$

On the other hand, the application of the *Sommerfeld integral representation* for the first term in (65) yields

$$H^{(1)}_{n-m}(k_r R_0)\mathrm{i}^{n-m}\mathrm{e}^{-\mathrm{i}(n-m)\phi_0} = \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}k_r\left(x\sqrt{1-p^2}-yp\right)}\left[\frac{\mathrm{e}^{\mathrm{i}(n-m)\sin^{-1}p}}{\sqrt{1-p^2}}\right]\mathrm{d}p. \tag{79}$$

In order to put this under the same kernel as the first term, we introduce a change of variable, $p = -\sin\phi_s$. Thus we establish the following expression for the first term in (65):

$$H^{(1)}_{n-m}(k_r R_0)\mathrm{i}^{n-m}\mathrm{e}^{-\mathrm{i}(n-m)\phi_0} = 2\int_{-\infty}^{\infty} \mathrm{d}s\,(L_s\mathrm{e}^{-\mathrm{i}(n-m)\phi_s})\mathrm{e}^{\mathrm{i}\vec{k_s}\cdot\vec{r}}. \tag{80}$$

Combining (77) and (80) in (65), we obtain $h_{n-m}(k_r r)$ as

$$h_{n-m}(k_r r) = 2\left(\sum_{s=-\infty}^{\infty} - \int_{-\infty}^{\infty}\mathrm{d}s\right)(L_s\mathrm{e}^{-\mathrm{i}(n-m)\phi_s})\mathrm{e}^{\mathrm{i}\vec{k_s}\cdot\vec{r}} \qquad x > 0 \tag{81a}$$

$$h_{n-m}(k_r r) = 2\left(\sum_{s=-\infty}^{\infty} - \int_{-\infty}^{\infty}\mathrm{d}s\right)\mathrm{e}^{\mathrm{i}k_r(x\cos\phi_s + y\sin\phi_s)}L_s\mathrm{e}^{-\mathrm{i}(n-m)\phi_s} \qquad x > 0 \tag{81b}$$

where s is an integer for the sum operation and a continuous variable for the integral; similarly for $x < 0$, we replace ϕ_s by $\pi - \phi_s$. In order to evaluate $\mathcal{H}_{n-m}(k_r d)$, we have first substituted (81) into (63), let $y \to 0$, and then take the limit as $x = \varepsilon \to +0$

$$\mathcal{H}_{n-m}(k_r d) = \lim_{\substack{r\to 0\\(y=0;x=\varepsilon\to 0)}} h_{n-m}(k_r r) \tag{82}$$

$$\mathcal{H}_{n-m}(k_r d) = 2\left\{\lim_{\varepsilon\to 0}\left(\sum_{s=-\infty}^{\infty} - \int_{-\infty}^{\infty}\mathrm{d}s\right)\mathrm{e}^{\mathrm{i}\varepsilon k_r\cos\phi_s}\right\}L_s\mathrm{e}^{-\mathrm{i}(n-m)\phi_s}. \tag{83}$$

In the expression above, we can define the *mode operator* $\mathcal{S}$ as

$$\mathcal{S} := \lim_{\varepsilon\to 0}\left(\sum_{s=-\infty}^{\infty} - \int_{-\infty}^{\infty}\mathrm{d}s\right)\mathrm{e}^{\mathrm{i}\varepsilon k_r\cos\phi_s}. \tag{84}$$

In terms of this *mode operator* $\mathcal{S}$, (83) can be written as

$$\mathcal{H}_{n-m}(k_r d) = 2\mathcal{S}L_s\exp[-\mathrm{i}(n-m)\phi_s] \tag{85a}$$

$$\mathcal{H}_{n-m}(k_r d) = 2\mathcal{S}L_s\exp[-\mathrm{i}(n-m)(\pi-\phi_s)] \tag{85b}$$

and the arithmetical means of (85a) and (85b) are given as

$$\mathcal{H}_{n-m}(k_r d) = \mathcal{S}L_s\{\exp[-\mathrm{i}(n-m)\phi_s] + \exp[-\mathrm{i}(n-m)(\pi-\phi_s)]\}. \tag{86}$$

4.3. Propagating range

In the application of the operator $\mathcal{S}$, we can split it into its *propagating* and *evanescent* ranges as

$$\mathcal{S} = \mathcal{S}_p + \mathcal{S}_e \tag{87}$$

where $\mathcal{S}_p$ indicates the propagating range of $\mathcal{S}$ corresponding to $|\sin\phi_s| < 1$, and $\mathcal{S}_e$ indicates the evanescent range of $\mathcal{S}$ corresponding to $|\sin\phi_s| > 1$. Thus, for the propagating range we have the *propagating mode operator* $\mathcal{S}_p$ as

$$\mathcal{S}_p = \sum_{s=-s_-}^{+s_+} - \int_{-\sigma_-}^{+\sigma_+}\mathrm{d}s. \tag{88}$$

The upper and lower limits in the integral above are given as

$$\sigma_{\mp} = \frac{k_r d}{2\pi}(1 \pm \sin\psi_i) \tag{89}$$

for $s_{\pm} \in Z$, and $\sigma_{\pm} \in R$. Differentiating the grating equation with respect to ϕ_s, we have established $\mathrm{d}s = \frac{\mathrm{d}\phi_s}{2\pi L_s}$, and using this in (88) we have obtained the *propagating mode operator* $\mathcal{S}_p$ as

$$\mathcal{S}_p = \sum_{s=-s_-}^{+s_+} -\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{\mathrm{d}\phi_s}{L_s}. \tag{90}$$

Inserting (87) into (86), we have

$$\mathcal{H}_{n-m}(k_r d) = \mathcal{S}L_s\{\exp[-\mathrm{i}(n-m)\phi_s] + \exp[-\mathrm{i}(n-m)(\pi-\phi_s)]\} \tag{91a}$$

$$\begin{aligned}\mathcal{H}_{n-m}(k_r d) = {}& \mathcal{S}_p L_s\{\exp[-\mathrm{i}(n-m)\phi_s] + \exp[-\mathrm{i}(n-m)(\pi-\phi_s)]\} \\ & + \mathcal{S}_e L_s\{\exp[-\mathrm{i}(n-m)\phi_s] + \exp[-\mathrm{i}(n-m)(\pi-\phi_s)]\}.\end{aligned} \tag{91b}$$

Defining

$$\begin{aligned}\mathcal{J}_{n-m}(k_r d) &= \mathcal{S}_p L_s\{\exp[-\mathrm{i}(n-m)\phi_s] + \exp[-\mathrm{i}(n-m)(\pi-\phi_s)]\} \\ &\equiv \sum_{s=1}^{\infty} J_{n-m}(sk_r d)[\exp(-\mathrm{i}sk_r d\sin\psi_i) + (-1)^{n-m}\exp(\mathrm{i}sk_r d\sin\psi_i)]\end{aligned} \tag{92a}$$

which corresponds to the *Bessel series* associated with the propagating range ($|\sin\phi_s| < 1$), and

$$\begin{aligned}\mathcal{N}_{n-m}(k_r d) &= -\mathrm{i}\mathcal{S}_e L_s\{\exp[-\mathrm{i}(n-m)\phi_s] + \exp[-\mathrm{i}(n-m)(\pi-\phi_s)]\} \\ &\equiv \sum_{s=1}^{\infty} N_{n-m}(sk_r d)[\exp(-\mathrm{i}sk_r d\sin\psi_i) + (-1)^{n-m}\exp(\mathrm{i}sk_r d\sin\psi_i)]\end{aligned} \tag{92b}$$

which corresponds to the *Neumann series* associated with the evanescent range ($|\sin\phi_s| > 1$), we can then express the *Schlömilch series* in terms of (92a) and (92b) as

$$\mathcal{H}_{n-m} = \mathcal{J}_{n-m} + \mathrm{i}\mathcal{N}_{n-m} \tag{93}$$

representing the overall range of all modes.

4.4. *Bessel series*

Substituting (90) into (92a), we have obtained the *Bessel series* associated with the propagating range ($|\sin\phi_s| < 1$) as

$$\mathcal{J}_n = \left[\sum_{s=-s_-}^{s_+} L_s\{\exp[-\mathrm{i}n\phi_s] + \exp[-\mathrm{i}n(\pi-\phi_s)]\}\right] - [1+(-1)^n]\left[\frac{\sin(n\pi/2)}{n\pi}\right]. \tag{94}$$

The evaluation of $n = 0$ requires special attention, and can be obtained from above as

$$\mathcal{J}_0 = \left[2\sum_{s=-s_-}^{s_+} L_s - 1\right]. \tag{95a}$$

For *even* n, from (94), we have

$$\mathcal{J}_{2n} = \left[2\sum_{s=-s_-}^{s_+} L_s \cos 2n\phi_s\right] \qquad \forall n \ni n \in N. \tag{95b}$$

Combining these two previous expressions in (95a) and (95b), we have obtained

$$\mathcal{J}_{2n} = \left[2\sum_{s=-s_-}^{s_+} L_s \cos 2n\phi_s - \delta_{n0}\right] \qquad \forall n \ni n \in Z_+. \tag{96}$$

For *odd n*, from (94), we have obtained

$$\mathcal{J}_{2n+1} = \left[-2\mathrm{i}\sum_{s=-s_-}^{s_+} L_s \sin(2n+1)\phi_s\right] \qquad \forall n \ni n \in Z_+ \tag{97}$$

where $Z_+ = \{0, 1, 2, 3, \ldots\}$.

4.5. Evanescent range

The evanescent mode operator $\mathcal{S}_e$ can be obtained from (87) as

$$\mathcal{S}_e = \mathcal{S} - \mathcal{S}_p. \tag{98}$$

In operating $\mathcal{S}_e$ on $\sigma(s)$, we first introduce the *Euler–Maclaurin summation formula* from Hardy [14] as

$$\left(\sum_{s=1}^{n} - \int_1^n \mathrm{d}s\right)\sigma(s) \approx \Omega + \frac{1}{2}\sigma(n) + \sum_{\xi=1}^{\infty}\frac{(-1)^{\xi-1}B_\xi}{(2\xi)!}\frac{\partial^{2\xi-1}\sigma(s)}{\partial s^{2\xi-1}}\bigg|_{s=n} \tag{99a}$$

where Ω is given as

$$\Omega = \left(\frac{1}{2} - \sum_{\xi=1}^{\infty}\frac{(-1)^{\xi-1}B_\xi}{(2\xi)!}\frac{\partial^{2\xi-1}}{\partial s^{2\xi-1}}\right)\sigma(s)|_{s=1} \tag{99b}$$

and B_ξ represents a *Bernoulli number*. Employing the *Euler–Maclaurin summation formula* of (99) in (98) effectively, we have finally obtained

$$\mathcal{S}_e\sigma(s) = -\left(\int_{\sigma_+}^{+s_++1} + \int_{-s_--1}^{-\sigma_-}\right)\sigma(s)\,\mathrm{d}s + \left(\frac{1}{2} - \sum_{\xi=1}^{n-1}\frac{(-1)^{\xi-1}B_\xi}{(2\xi)!}\frac{\partial^{2\xi-1}}{\partial s^{2\xi-1}}\right)\sigma(s)|_{s=\pm s_\pm\pm 1} + \mathcal{R}_n \tag{100a}$$

where $\mathcal{R}_n$ corresponds to the remainder term and is given as

$$\mathcal{R}_n = \sum_{\xi=n}^{\infty}\frac{(-1)^{\xi}B_\xi}{(2\xi)!}\frac{\partial^{2\xi-1}\sigma(s)}{\partial s^{2\xi-1}}\bigg|_{s=s_-+1}^{s=s_++1} \tag{100b}$$

$s = \pm s_\pm \pm 1$ are the first non-propagating modes and $\pm\frac{\pi}{2} \mp \mathrm{i}\eta_s^\pm$ are the corresponding values of ϕ; the B_ξ are *Bernoulli numbers*; $s = \pm s_\pm$ are the *grazing modes* or *Rayleigh values*, and $\pm\frac{\pi}{2}$ are the corresponding values of ϕ. By direct application of the chain rule of differentiation, we can write

$$\frac{\partial^{2\xi-1}\sigma(s)}{\partial s^{2\xi-1}} = \frac{1}{\Delta^{2\xi-1}}\frac{\partial^{2\xi-1}\sigma(\sin\phi_s)}{\partial(\sin\phi_s)^{2\xi-1}} \tag{101}$$

where $\Delta \equiv k_r d/2\pi$, and introducing the relationship between *Bernoulli polynomial* and *Bernoulli numbers* as

$$B_{2\xi}(0) \equiv (-1)^{\xi-1}B_\xi \tag{102}$$

we can express the *evanescent mode operator* $\mathcal{S}_e$ in its most desired form as

$$\mathcal{S}_e\sigma(\sin\phi_s) = -\left(\int_{\frac{\pi}{2}}^{\frac{\pi}{2}-\mathrm{i}\eta_s^+} + \int_{-\frac{\pi}{2}+\mathrm{i}\eta_s^-}^{-\frac{\pi}{2}}\right)\frac{\sigma(\sin\phi_s)\,\mathrm{d}\phi_s}{2\pi L_s}$$
$$+\left[\frac{1}{2} - \sum_{\xi=1}^{n-1}\frac{B_{2\xi}(0)}{(2\xi)!\Delta^{2\xi-1}}\frac{\partial^{2\xi-1}\sigma(\sin\phi_s)}{\partial(\sin\phi_s)^{2\xi-1}}\right]\bigg|_{s=\pm s_\pm\pm 1} + \mathcal{R}_n \tag{103a}$$

where $\mathcal{R}_n$ represents the remainder and is given as

$$\mathcal{R}_n = \sum_{\xi=n}^{\infty} \frac{(-1)^{\xi} B_{\xi}}{(2\xi)!\Delta^{2\xi-1}} \frac{\partial^{2\xi-1}\sigma(\sin\phi_s)}{\partial(\sin\phi_s)^{2\xi-1}}\Bigg|_{s=s_-+1}^{s=s_++1}. \tag{103b}$$

4.6. Neumann series

Substituting (103a) into (92b), we have obtained the *Neumann series* associated with the evanescent range ($|\sin\phi_s| > 1$) as

$$\begin{aligned}\mathcal{N}_{2n} = \frac{1}{n\pi} + \frac{1}{\pi}\Bigg\{ &\sum_{m=1}^{n}\left[\frac{(-1)^m 2^{2m}(n+m-1)!}{(2m)!(n-m)!}\right]\frac{B_{2m}(\Delta\sin\psi_i)}{\Delta^{2m}} \\ &-\left(\sum_{s=-s_-}^{-1} - \sum_{s=0}^{s_+}\right)\sum_{m=1}^{n}\left[\frac{(-1)^m 2^{2m-1}(n+m-1)!}{(2m-1)!(n-m)!}\right]\frac{(s+\Delta\sin\psi_i)^{2m-1}}{\Delta^{2m}}\Bigg\} \\ &+\frac{(-1)^{n+1}}{\pi\Delta}\left\{\sum_{s=s_++1}^{\infty}\frac{\left[\left(\frac{S}{\Delta}+\sin\psi_i\right)-\sqrt{\left(\frac{S}{\Delta}+\sin\psi_i\right)^2-1}\right]^{2n}}{\sqrt{\left(\frac{S}{\Delta}+\sin\psi_i\right)^2-1}}\right. \\ &\left.+\sum_{s=s_-+1}^{\infty}\frac{\left[\left(\frac{S}{\Delta}-\sin\psi_i\right)-\sqrt{\left(\frac{S}{\Delta}-\sin\psi_i\right)^2-1}\right]^{2n}}{\sqrt{\left(\frac{S}{\Delta}-\sin\psi_i\right)^2-1}}\right\}\end{aligned} \tag{104a}$$

$\forall n \ni n \in N$, and

$$\begin{aligned}\mathcal{N}_{2n+1} = \frac{1}{\mathrm{i}\pi}\Bigg\{ &2\sum_{m=0}^{n}\left[\frac{(-1)^m 2^{2m}(n+m)!}{(2m+1)!(n-m)!}\right]\frac{B_{2m+1}(\Delta\sin\psi_i)}{\Delta^{2m+1}} \\ &-\left(\sum_{s=-s_-}^{-1} - \sum_{s=0}^{s_+}\right)\sum_{m=0}^{n}\left[\frac{(-1)^m 2^{2m}(n+m)!}{(2m)!(n-m)!}\right]\frac{(s+\Delta\sin\psi_i)^{2m}}{\Delta^{2m+1}}\Bigg\} \\ &+\frac{(-1)^{n+1}}{\mathrm{i}\pi\Delta}\left\{\sum_{s=s_++1}^{\infty}\frac{\left[\left(\frac{S}{\Delta}+\sin\psi_i\right)-\sqrt{\left(\frac{S}{\Delta}+\sin\psi_i\right)^2-1}\right]^{2n+1}}{\sqrt{\left(\frac{S}{\Delta}+\sin\psi_i\right)^2-1}}\right. \\ &\left.-\sum_{s=s_-+1}^{\infty}\frac{\left[\left(\frac{S}{\Delta}-\sin\psi_i\right)-\sqrt{\left(\frac{S}{\Delta}-\sin\psi_i\right)^2-1}\right]^{2n+1}}{\sqrt{\left(\frac{S}{\Delta}-\sin\psi_i\right)^2-1}}\right\}\end{aligned} \tag{104b}$$

$\forall n \ni n \in Z_+$, where $Z_+ = \{0, 1, 2, 3, \ldots\}$.

For the evaluation of the special case with $n = 0$, we have

$$\mathcal{N}_0 = -\mathrm{i}\left[2\left(\sum_{s=-\infty}^{\infty} - \int_{-\infty}^{\infty}\mathrm{d}s\right)L_s + 1 - \sum_{s=-s_-}^{s_+} L_s\right] \tag{105a}$$

$$\begin{aligned}\mathcal{N}_0 = &\left[-\mathrm{i}\left(1 - 2\int_{-\infty}^{\infty}\mathrm{d}s\, L_s\right) - \frac{2}{\pi}\sum_{s=1}^{\infty}\left(\frac{1}{s}\right)\right] + \left[\frac{1}{\pi}\left(\sum_{s=1}^{s_+} + \sum_{s=1}^{s_-}\right)\frac{1}{s}\right] \\ &-\frac{1}{\pi}\left\{\sum_{s=s_++1}^{\infty}\left[\frac{1}{\sqrt{(s+\Delta\sin\psi_i)^2-\Delta^2}} - \frac{1}{s}\right]\right. \\ &\left.+\sum_{s=s_-+1}^{\infty}\left[\frac{1}{\sqrt{(s-\Delta\sin\psi_i)^2-\Delta^2}} - \frac{1}{s}\right]\right\}.\end{aligned} \tag{105b}$$

The evaluation of the first term in (105b), namely

$$\left[-\mathrm{i}\left(1-2\int_{-\infty}^{\infty}\mathrm{d}s\,L_s\right)-\frac{2}{\pi}\sum_{s=1}^{\infty}\left(\frac{1}{s}\right)\right] \tag{106}$$

requires special attention, and should be dealt with separately. Introducing a change of variables as $\omega = \Delta\sin\bar{\omega}$, we establish that

$$2\int_{-\infty}^{\infty}\mathrm{d}s\,L_s = \frac{1}{\pi}+\frac{2}{\mathrm{i}\pi}\int_{\Delta}^{\infty}\frac{\mathrm{d}\bar{\omega}}{\sqrt{\bar{\omega}^2-\Delta^2}} \tag{107a}$$

$$2\int_{-\infty}^{\infty}\mathrm{d}s\,L_s = 1+\frac{2}{\mathrm{i}\pi}\lim_{\zeta\to\infty}\left[\ln\left(\frac{2\zeta}{\Delta}\right)\right]. \tag{107b}$$

In an attempt to evaluate the limit above, we introduced *Euler's constant* as $\gamma = 1.781\ldots$ into (107b) and obtained

$$2\int_{-\infty}^{\infty}\mathrm{d}s\,L_s = 1+\frac{2}{\mathrm{i}\pi}\left[\sum_{s=1}^{\infty}\left(\frac{1}{s}\right)-\ln\left(\gamma\frac{k_r d}{4\pi}\right)\right]. \tag{108}$$

Inserting the result of (108) into (106), we have obtained

$$\left[-\mathrm{i}\left(1-2\int_{-\infty}^{\infty}\mathrm{d}s\,L_s\right)-\frac{2}{\pi}\sum_{s=1}^{\infty}\left(\frac{1}{s}\right)\right] = -\left(\frac{2}{\pi}\right)\ln\left(\frac{\Delta\gamma}{2}\right). \tag{109}$$

Upon inserting this result of (109) into (105b) we finally obtained the generalized $\mathcal{N}_0$ for the oblique incidence case as

$$\mathcal{N}_0 = -\left(\frac{2}{\pi}\right)\ln\left(\frac{\Delta\gamma}{2}\right)+\left[\frac{1}{\pi}\left(\sum_{s=1}^{s_+}+\sum_{s=1}^{s_-}\right)\frac{1}{s}\right]-\frac{1}{\pi}\left\{\sum_{s=s_++1}^{\infty}\left[\frac{1}{\sqrt{(s+\Delta\sin\psi_i)^2-\Delta^2}}-\frac{1}{s}\right]\right.$$
$$\left.+\sum_{s=s_-+1}^{\infty}\left[\frac{1}{\sqrt{(s-\Delta\sin\psi_i)^2-\Delta^2}}-\frac{1}{s}\right]\right\}. \tag{110}$$

One should read the (−) sign in the second term of Twersky's expression for the non-oblique incidence as (+).

5. Conclusion

In this paper, the exact analytical expressions for the electric and magnetic field intensities associated with the classical electromagnetic scattering problem of *transverse electric multiple scattering by an infinite grating of infinitely long circular dielectric cylinders to obliquely incident plane electromagnetic waves* has been derived in terms of the elementary function representations of *Schlömilch series* introduced by Twersky [1]. It has been demonstrated that the elementary function representations of *Schlömilch series* can effectively be employed to determine the exact analytical solution of the classical electromagnetic scattering problem by an infinite array of insulating circular dielectric cylinders for the obliquely incident transverse electric fields. Our expressions for the elementary function representations of *Schlömilch series* associated with *obliquely incident plane electromagnetic waves* are generalizations of those presented by Twersky [1], and our solution reduces to his for the non-oblique incident case.

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